Line graphs, triangle graphs, and further generalizations

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Outline

1. Line graphs, k-line graphs
2. Characterization: $G \square F$ is a triangle graph
3. Characterization: $G \square F$ is a k-line graph
4. Anti-Gallai graphs, k-anti-Gallai-graphs
5. Recognizing k-line graphs and k-anti-Gallai graphs: NP-complete for every $k \geq 3$

- Line graph $L(G)$

  vertices of $L(G)$ $\leftrightarrow$ edges (K_2-subgraphs) of G

  edges of $L(G)$ $\leftrightarrow$ two edges share a K_1 in G

- Triangle graph $T(G)$

  vertices of $T(G)$ $\leftrightarrow$ triangles (K_3-subgraphs) of G

  edges of $T(G)$ $\leftrightarrow$ two triangles share an edge (K_2) in G

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- k-line graph $L_k(G)$

  vertices of $L_k(G)$ $\leftrightarrow$ K_k-subgraphs of G

  edges of $L_k(G)$ $\leftrightarrow$ two K_k-subgraphs share $\boldsymbol{k-1}$ vertices (K_{k-1}) in G

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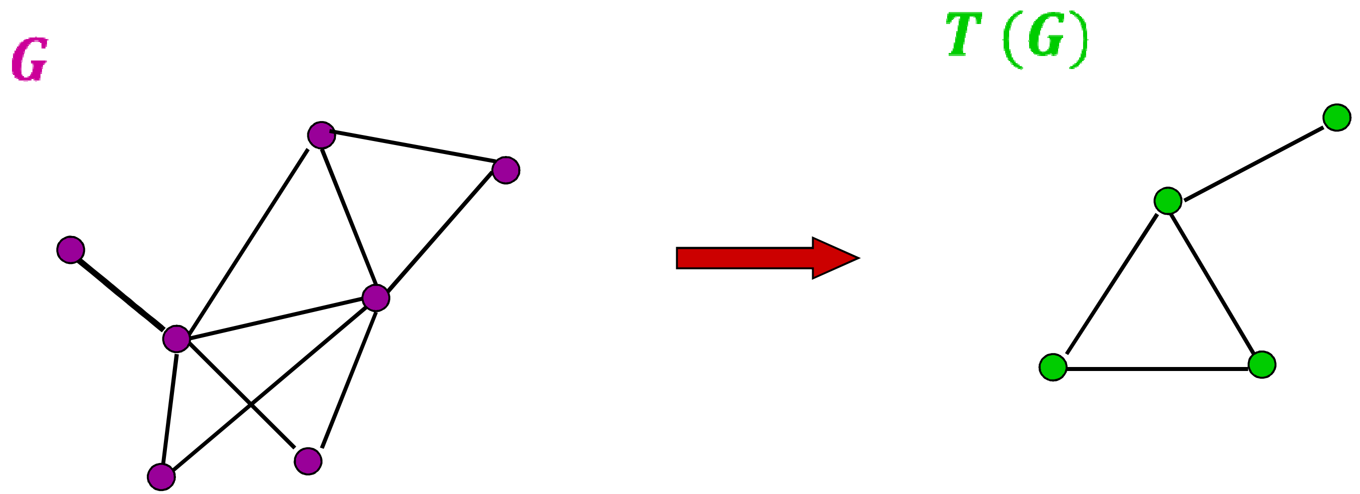
$$L_1(G) = K_{|V(G)|},\ \ L_2(G) = L(G),\ \ L_3(G) = T(G)$$

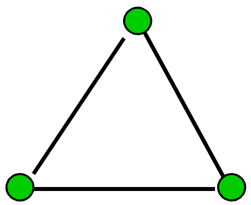
Triangle graph



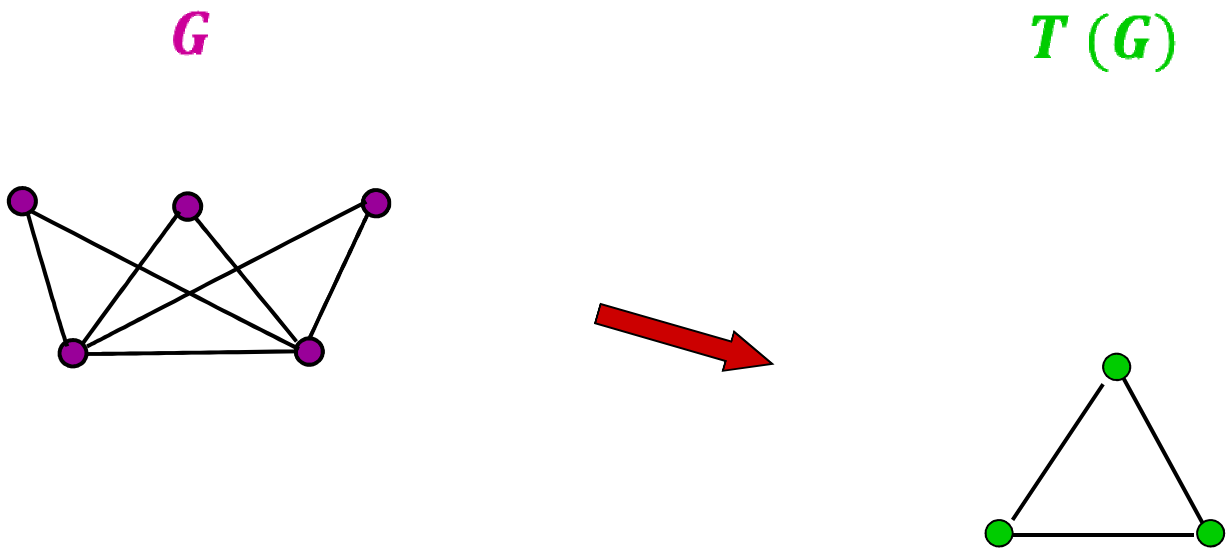
Triangle graph

G

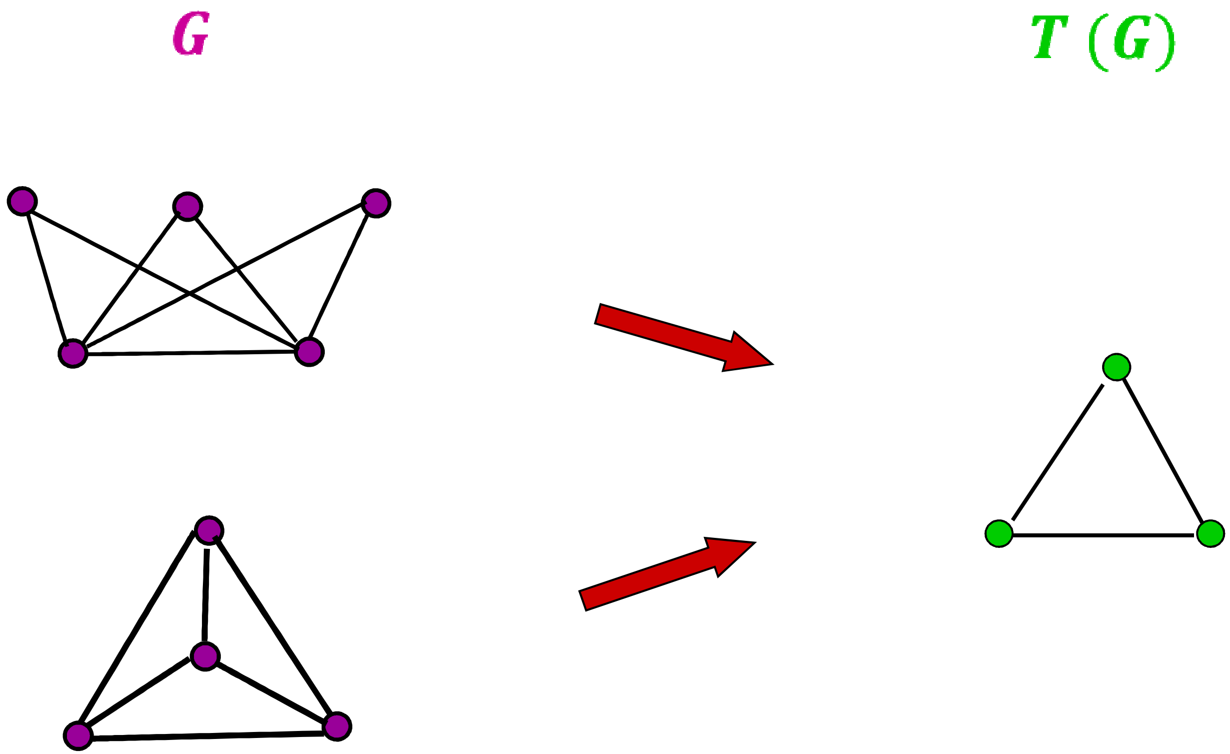
$T(G)$



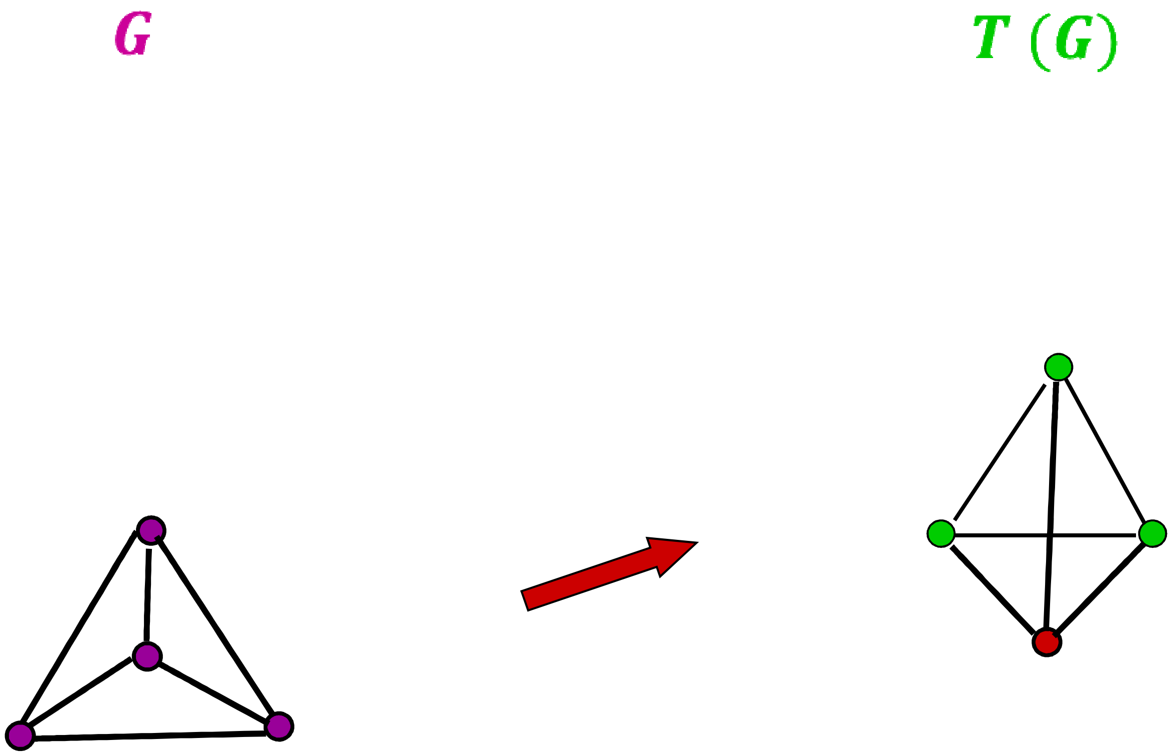
Triangle graph



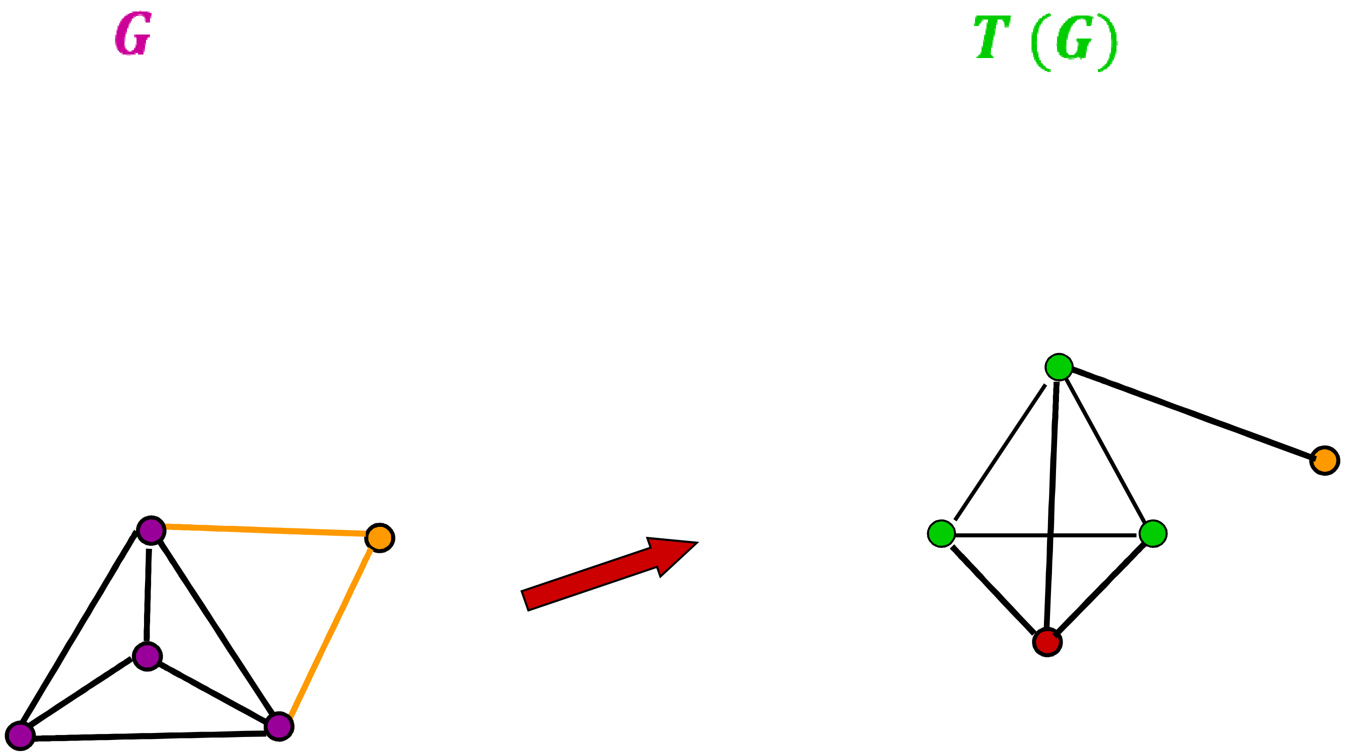
Triangle graph



Triangle graph

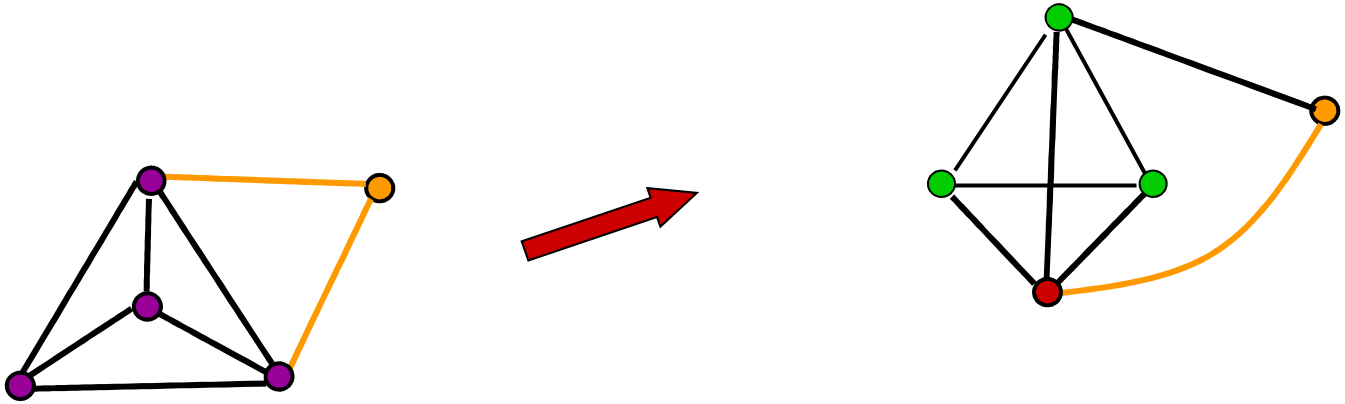


Triangle graph



Triangle graph

G $T(G)$



Triangle graph – some observations

$G = T(G')$ $\Rightarrow$

- G is $K_{1,4}$-free
- If G contains a K_4 $v_1v_2v_3v_4$ with a preimage K_4 in G', then each further vertex of G is adjacent to 0 or 2 vertices from $v_1v_2v_3v_4$
- If G contains a diamond D, then there is a further vertex adjacent to exactly three vertices of D

$G = T(G')$ & G has no K_4-component:

G' is K_4-free $\Leftrightarrow$ G is diamond-free

k-line graphs – analogous observations

$G = L_k(G')$ $\Rightarrow$

- G is $K_{1,k+1}$-free
- If G contains a K_{k+1} subgraph F with a preimage K_{k+1} in G', then each further vertex of G is adjacent to 0 or 2 vertices of F
- If G contains a diamond D, then there are further $k-2$ vertices of G forming K_{k+1} together with three vertices of D

$G = L_k(G')$ & G has no K_{k+1}-component (k $\geq$ 2) :

G' is K_{k+1}-free $\Leftrightarrow$ G is diamond-free

Cartesian product

$G\square F$:

- $V(G\square F) = V(G) \times V(F)$,
- $(v, u), (v', u')$ are adjacent iff

$$[v = v' \ \& \ uu' \in E(F)] \text{ or } [vv' \in E(G) \ \& \ u = u']$$

Cartesian product – as a triangle graph

(G and F are connected, non-edgeless graphs)

$G \square F$ is a triangle graph if and only if
$F = K_n$ and G is the line graph of a K_3-free graph (or vice versa).

Proof.

1. If G and F are non-complete $\rightarrow \exists$ induced paths $u_1u_2u_3$ and $v_1v_2v_3$
$\rightarrow (u_2, v_2)$ has four independent neighbors $\rightarrow G \square F$ is not a triangle-graph

2. $G \square F = T(H) \rightarrow G, F$ claw-free and diamond-free $\rightarrow G, F$ are line graphs

3. $G = L(G')$
if $G = K_3$, then let $G' = K_{1\ 3}$
if $G \neq K_3$, G is diamond-free $\rightarrow G'$ is K_3-free

4. Sufficiency: $G = L(G')$ & G' is K_3-free $\rightarrow T(G' \vee nK_1) = L(G') \square K_n = G \square K_n$

Cartesian products and k-line graphs

(G and F are non-edgeless connected graphs)

$G \square F$ is a $\boldsymbol{k}$-line graph if and only if $\exists\ a, b$

- G is the $\boldsymbol{a}$-line graph of a K_{a+1}-free graph,
- F is the $\boldsymbol{b}$-line graph of a K_{b+1}-free graph, and
- $\boldsymbol{a} + \boldsymbol{b} \leq \boldsymbol{k}$

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Sufficiency:

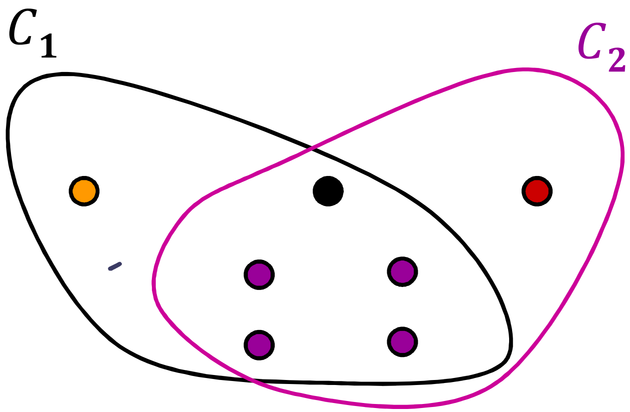
- Let $G = L_{\boldsymbol{a}}(G')$, $F = L_{\boldsymbol{b}}(F')$, G' is K_{a+1}-free , F' is K_{b+1}-free

  $\Longrightarrow \quad L_{\boldsymbol{a}+\boldsymbol{b}}(G' \vee F') = G \square F$
- If $k - (a + b) = c > 0$, then $L_{\boldsymbol{k}}\left((G' \vee F') \vee K_{\boldsymbol{c}}\right) = G \square F$

Cartesian products and k-line graphs

Necessity:

- Lemma1: $H = L_k(H')$ and H contains an induced 4-cycle: $c_1c_2c_3c_4$ $\longrightarrow$ for the preimages in H': $C_1 \setminus C_2 = C_4 \setminus C_3$.



Cartesian products and k-line graphs

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- Lemma 1: $H = L_{\boldsymbol{k}}(H')$ and H contains an induced C_4: $c_1c_2c_3c_4$ ⟶
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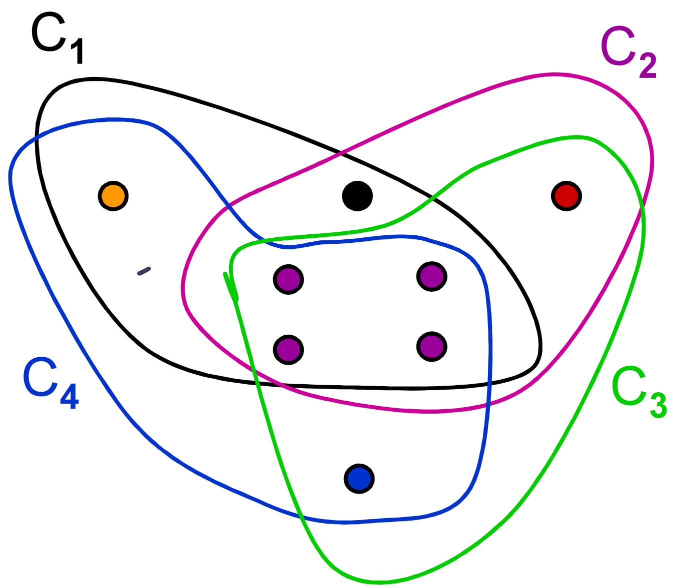

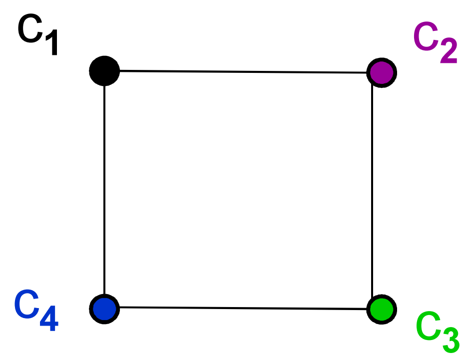

Cartesian products and k-line graphs

Necessity:

Let $H = G\square F = L_{\boldsymbol{k}}(H')$.
For a copy $G_{\boldsymbol{j}}$ of G consider the preimage k-cliques $C(u_{\boldsymbol{i}}, v_{\boldsymbol{j}})$ of H', and define the sets of the universal and non-universal vertices:
$U_{\boldsymbol{j}}{}^{G} = \bigcap\{C(v_{i}, u_{\boldsymbol{j}}){:}\ v_{\boldsymbol{i}} \in V(G)\}$, $X_{\boldsymbol{j}}{}^{G} = \bigcup\{C(v_{i}, u_{\boldsymbol{j}}){:}\ v_{\boldsymbol{i}} \in V(G)\} \setminus U_{\boldsymbol{j}}{}^{G}$

Cartesian products and k-line graphs

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Lemma 2:

- The non-universal vertices are the same: $X_j{}^G = X_m{}^G$
- If $v_i v_m \in E(F) \rightarrow \exists\, z, w \quad U_m{}^G = U_i{}^G \setminus \{z\} \cup \{w\}$

Cartesian products and k-line graphs

Necessity:

Let $H = G\square F = L_k(H')$.
For a copy G_j of G consider the preimage k-cliques $C(u_i, v_j)$ of H', and define the sets of the universal and non-universal vertices:
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- If $v_i v_m \in E(F) \rightarrow \exists\, z, w \quad U_m{}^G = U_i{}^G \setminus \{z\} \cup \{w\}$

To complete the proof:
$x = |U_1{}^G|$ and $y = |U_1{}^F| \rightarrow$

- G is the $(k-x)$-line graph of a K_{k-x+1}-free graph, and
- F is the $(k-y)$-line graph of a K_{k-y+1}- free graph.
- $x + y \geq k$

Subgraphs of k-line graphs

- Anti-Gallai graph $\Delta(G)$

  vertices of $\Delta(G)$ $\leftrightarrow$ edges (K_2-subgraphs) of G

  edges of $\Delta(G)$ $\leftrightarrow$ the two edges (share a K_1 and) are in a common K_3 in G

- k-anti-Gallai graph $\Delta_k(G)$

  vertices of $\Delta_k(G)$ $\leftrightarrow$ K_k-subgraphs of G

  edges of $\Delta_k(G)$ $\leftrightarrow$ two K_k-subgraphs (share a K_{k-1} and) contained in a common K_{k+1} in G

- k-Gallai graph $\Gamma_k(G)$

  vertices of $\Gamma_k(G)$ $\leftrightarrow$ K_k-subgraphs of G

  edges of $\Gamma_k(G)$ $\leftrightarrow$ two K_k-subgraphs share a K_{k-1} and NOT contained in a common K_{k+1} in G

- $\Delta_1(G) = G$, $\Delta_2(G) = \Delta(G)$; $\Gamma_1(G) = \bar{G}$, $\Gamma_2(G) = \Gamma(G)$

Recognizing triangle graphs

The following problems are NP-complete:

- Recognizing triangle graphs
- Deciding whether a given graph is the triangle graph of a K_4-free graph

Anand, Escuardo, Gera, Hartke, Stolee (2012):

- Deciding whether a given connected graph is the anti-Gallai graph of a K_4-free graph --- NP-complete problem

Recognizing triangle graphs

The following problems are NP-complete:
- Recognizing triangle graphs
- Deciding whether a given graph is the triangle graph of a K_4-free graph

Proof:

Lemma1: If $F = \Delta(F')$ and F is connected, then

F' is K_4-free ⟺ (1) every maximal clique of F is a triangle and any two triangles share at most one vertex

Lemma2: If $G = T(G')$ and G is connected, $G \neq K_4$, then

G' is K_4-free ⟺ (2) each vertex of G is contained in at most three maximal cliques, and these are edge-disjoint

- Given a connected instance F

$\rightarrow$ Check (1)

$\rightarrow$ If it holds, construct the clique graph $G = \boldsymbol{K}(F)$ $\rightarrow$ G satisfies (2) + conn

$\rightarrow$ $F = \Delta(H)$ ⟺ $G = T(H)$

(suppose: H is 'triangle-restricted')

Recognizing k-line graphs

- For $k = 1 \rightarrow$ trivial
- For $k = 2 \rightarrow$ polynomial-time (Beineke), linear-time (Lehot; Roussopoulos)
- For $k = 3 \rightarrow$ we have proved: NP-complete

The following problems are NP-complete for each $k \geq 4$

- Recognizing k-line graphs
- Deciding whether a given graph is the k-line graph of a K_{k+1}-free graph

Proof:

- $k = 3 \rightarrow$ NP-complete
- Induction on k:

G is the k-line graph of a K_{k+1}-free graph $\leftrightarrow$

$G \square K_2$ is the $(k+1)$-line graph of a K_{k+2}-free graph

(on the class of connected graphs)

Recognizing k-anti-Gallai graphs

- For $k = 1 \rightarrow$ trivial
- For $k = 2 \rightarrow$ NP-complete (Anand et al.)

The following problems are NP-complete for each $k \geq 3$

- Recognizing k-anti-Gallai graphs
- Deciding whether a given graph is the k-anti-Gallai graph of a K_{k+2}-free graph

Recognizing k-anti-Gallai graphs

Proof:

Lemma: G is K_{k+2}-free $\leftrightarrow$ $\Delta_{\boldsymbol{k}}(G)$ is diamond-free $\leftrightarrow$ each maximal clique of $\Delta_{\boldsymbol{k}}(G)$ is either an isolated vertex or a $(k+1)$-clique, and any two maximal cliques intersect in at most one vertex.

- Induction on k
- Given a connected instance G

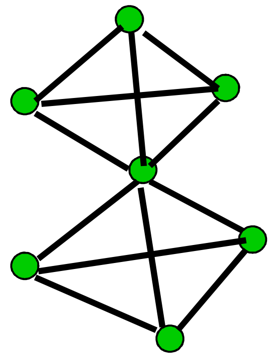
$\rightarrow$ Check diamond-freeness

$\rightarrow$ If it holds, construct G^*

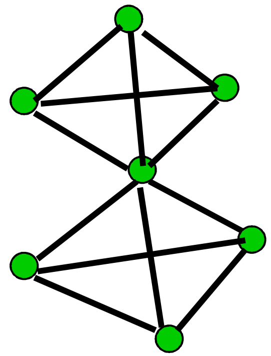
$\rightarrow$ $G = \Delta_{\boldsymbol{k}}(G') \leftrightarrow$

$$\mathrm{G}^* = \Delta_{k+1}(G' \vee 2K_{\mathbf{1}})$$

G1



G2



Recognizing k-anti-Gallai graphs

Proof:

Lemma: G is K_{k+2}-free $\leftrightarrow$ $\Delta_k(G)$ is diamond-free $\leftrightarrow$ each maximal clique of $\Delta_k(G)$ is either an isolated vertex or a $(k+1)$-clique, and any two maximal cliques intersect in at most one vertex.

- Induction on k
- Given a connected instance G

$\longrightarrow$ Check diamond-freeness

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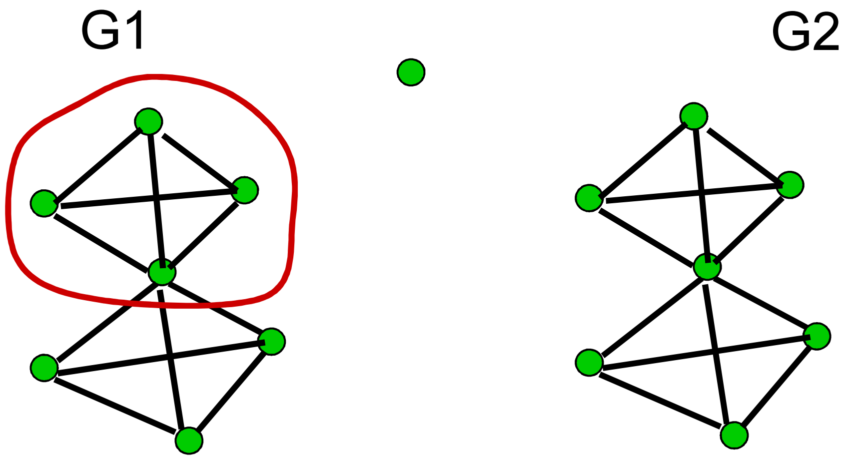
$\longrightarrow$ $G = \Delta_k(G')$ $\leftrightarrow$

$G^* = \Delta_{k+1}(G' \vee 2K_1)$



Recognizing k-anti-Gallai graphs

Proof:

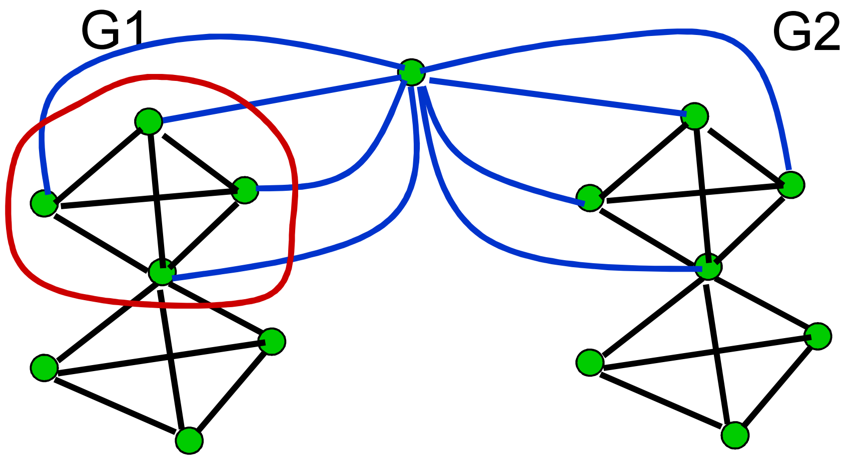
Lemma: G is K_{k+2}-free $\leftrightarrow$ $\Delta_k(G)$ is diamond-free $\leftrightarrow$
each maximal clique of $\Delta_k(G)$ is either an isolated vertex or a $(k+1)$-clique, and any two maximal cliques intersect in at most one vertex.

- Induction on k
- Given a connected instance G

→ Check diamond-freeness

→ If it holds, construct G*

→ G = Δ_k(G') $\leftrightarrow$ G*=Δ_{k+1}(G' V $2K_1$)



Recognizing k-anti-Gallai graphs

Proof:

Lemma: G is K_{k+2}-free ↔ Δ_k(G) is diamond-free

↔ each maximal clique of Δ_k(G) is either an isolated vertex or a (k+1)-clique, and any two maximal cliques intersect in at most one vertex.

- Induction on k
- Given a connected instance G

→ Check diamond-freeness

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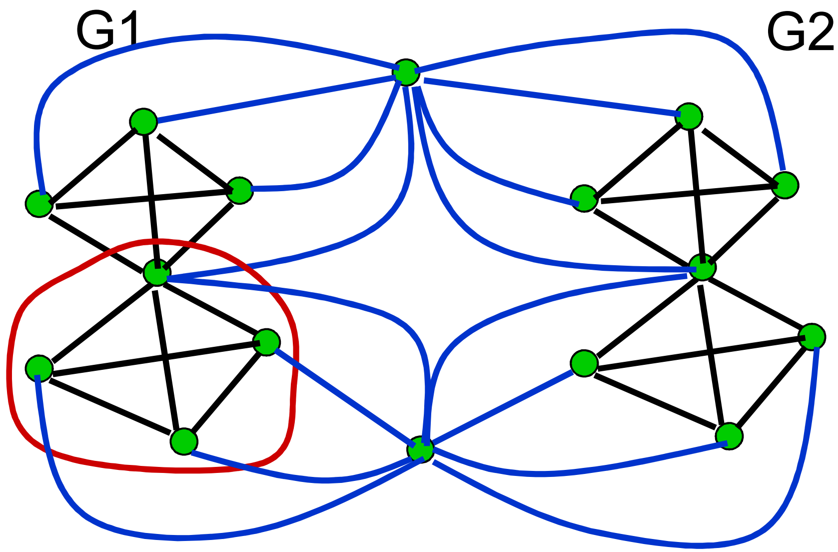

Recognizing k-anti-Gallai graphs

Proof:

Lemma: G is K_{k+2}-free $\leftrightarrow$ $\Delta_k(G)$ is diamond-free

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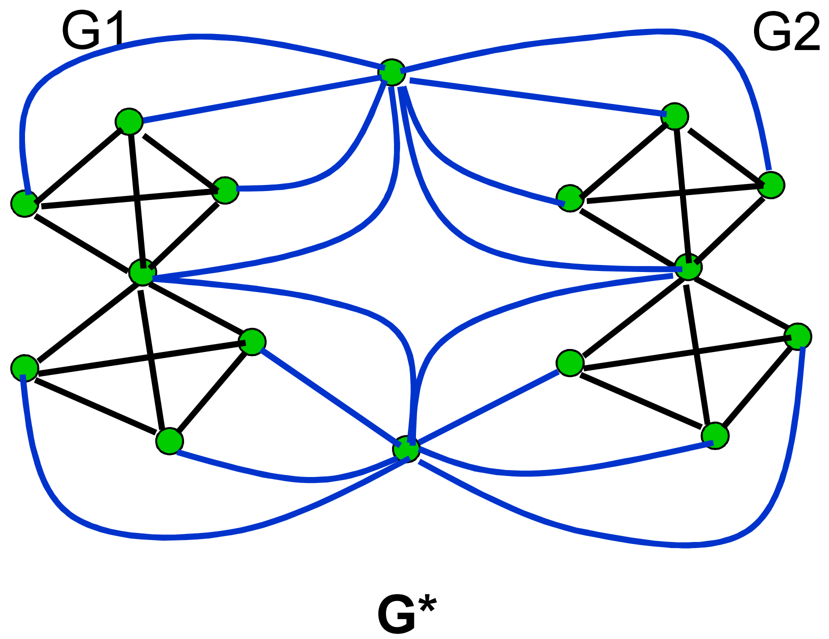
- Induction on k
- Given a connected instance G

$\rightarrow$ Check diamond-freeness

$\rightarrow$ If it holds, construct G*

$\rightarrow$ $G = \Delta_k(G')$ $\leftrightarrow$ $G^* = \Delta_{k+1}(G' \vee 2K_1)$



Recognition problem of k-Gallai graphs

OPEN for each $k \geq 2$:
What is the time complexity of the recognition problem of k-Gallai graphs?

k-Gallai graph $\Gamma_{k}(G)$
vertices of $\Gamma_{k}(G) \leftrightarrow$ K_{k}-subgraphs of G
edges of $\Gamma_{k}(G) \leftrightarrow$ two K_{k}-subgraphs share a K_{k-1} and NOT contained in a common K_{k+1} in G

Remark.
Deciding whether a given graph is the k-Gallai graph of a K_{k+1}-free graph $\rightarrow$ NP-complete (for each $k \geq 3$)
But we have no polynomial-time checkable property *P* with:
$\Gamma(G')$ has property *P* if and only if G' is K_{k+1}-free

Thank you for your attention…